Theory and Methodology

Possibilistic linear regression analysis for fuzzy data

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Abstract: Fuzzy data given by expert knowledge can be regarded as a possibility distribution by which possibilistic linear systems are defined. Recently, it has become important to deal with fuzzy data in connection with expert knowledge. Three formulations of possibilistic linear regression analysis are proposed here to deal with fuzzy data. Since our formulations can be reduced to linear programming problems, the merit of our formulations is to be able to obtain easily fuzzy parameters in possibilistic linear models and to add other constraint conditions which might be obtained from expert knowledge of fuzzy parameters. This approach can be regarded as a fuzzy interval analysis in a fuzzy environment.

Keywords: Fuzzy sets, regression, possibilistic linear systems, fuzzy data

1. Introduction

Recently, it has become important to deal with fuzzy data originated from a fuzzy phenomenon. A model of such a fuzzy phenomenon might be represented as a fuzzy system equation described by possibility measures. Since fuzzy data defined by fuzzy numbers can be regarded as possibility distributions, a possibilistic linear system is employed as a model of fuzzy phenomenon. A possibilistic linear system is defined by fuzzy parameters which mean possibility distributions [1]. In the conventional regression model, deviations between the observed and the estimated values are supposed to be due to measurement errors. Here, these deviations are assumed to depend on the fluctuations of the system parameters. Thus these deviations are reflected in a possibilistic linear system.

Fuzzy linear regression analysis has been discussed in [2]-[7]. Fuzzy models constructed by the expert knowledge “If A, then B” have been discussed in [8]. The aim of these papers is to obtain a fuzzy analysis fitting to a fuzzy phenomenon.

In the previous paper [6], we have discussed two types of possibilistic linear regression analysis which are called the Min problem and the Max problem. In this paper, we propose another formulation called Conjunction problem and discuss the mutual relations of three formulations to clarify the properties of fuzzy data analysis. Our method for obtaining possibilistic linear models can be reduced to linear programming problems so that the merit of these formulations is to be able to solve these problems by linear programming and also to introduce other constraint conditions with
respect to fuzzy parameters, which will be obtained from expert knowledge.

Our main concerns are on unifying formulations and their properties to be able to use possibilistic linear regression analysis. Some numerical examples are shown to illustrate three formulations of possibilistic linear regression analysis.

2. Possibilistic linear systems

A linear system whose parameters are defined by possibility distributions is called a possibilistic linear system. Here, a possibility distribution is represented by a fuzzy number \( A \) which satisfies

(i) \[ \mu_A(a) \] \( \in [0,1] \),
(ii) there exists an \( a \) such that \( \mu_A(a) = 1 \),
(iii) a fuzzy number is convex, i.e. \( \forall \lambda \in [0,1] \):
\[
\mu_A(\lambda a_1 + (1 - \lambda) a_2) \geq \min(\mu_A(a_1), \mu_A(a_2)),
\]
where \( \mu_A(a) \) is a membership function of the fuzzy number \( A \) and \( \mu \) denotes minimum.

According to the notations of L-R fuzzy numbers defined by Dubois and Prade [1], let us define only a symmetrical fuzzy number \( A_i \).

**Definition 2.1.** A symmetrical fuzzy number \( A_i \) denoted as \( A_i = (\alpha_i, c_i) \) is defined as
\[
\mu_{A_i}(a_i) = L((a_i - \alpha_i)/c_i), \quad c_i > 0,
\]
where \( \alpha_i \) is a center, \( c_i \) is a spread, and \( L(a_i) \) is a shape function of fuzzy number defined by:

(i) \( L(a_i) = L(-a_i) \),
(ii) \( L(0) = 1 \),
(iii) \( L(a_i) \) is strictly decreasing function for \( a_i > 0 \),
(iv) \( \{a_i | L(a_i) \geq 0\} \) is a closed interval.

In the examples of this paper, we use \( L(a) = \max(0, 1 - |a|) \) shown in Figure 1. Note that a fuzzy number \( A_i \) is reduced to a conventional number \( a_i \), when \( c_i = 0 \).

Let us consider a function \( f(x, a) \) where \( x = (x_1, \ldots, x_n)^t \) and \( a = (a_1, \ldots, a_n) \). If parameters are given by a fuzzy number vector \( A = (A_1, \ldots, A_n) \), the function is called a possibilistic function denoted by \( f(x, A) \). Given \( x \), the fuzzy number \( Y = f(x, A) \) can be defined by the following extension principle [9].

**Definition 2.2.** The membership function of \( Y \) is defined as
\[
\mu_Y(y) = \sup \{ \mu_A(a_1) \cdots \mu_A(a_n), \quad \{a_i | y = f(x, a)\},
\]
where
\[
\mu_Y(y) = 0 \quad \text{if} \quad \{a_i | y = f(x, a)\} = \emptyset.
\]

Let us consider a possibilistic linear system
\[
Y = A_1 x_1 + \cdots + A_n x_n = A x,
\]
where \( A_i = (\alpha_i, c_i) \) and \( x_i \) is a real number.

**Theorem 2.1.** The fuzzy output \( Y \) in (2.3) can be represented as
\[
Y = (ax, c | x) \in L,
\]
where \( a = (\alpha_1, \ldots, \alpha_n), c = (c_1, \ldots, c_n) \) and \( |x| = (|x_1|, \ldots, |x_n|)^t \).

The proof of Theorem 2.1 is shown in [6]. Given the possibility distributions of parameter \( A \), the possibility distribution of output \( Y \) can be easily calculated by Theorem 2.1. A possibilistic linear system can be symbolically rewritten as
\[
(\alpha_1, c_1 \in L) x_1 + \cdots + (\alpha_n, c_n) x_n = (ax, c | x) \in L.
\]

In contrast to the above, a probabilistic linear system whose parameters are independent Gaussian random variables can be symbolically written as
\[
N(\gamma_1, \sigma_1^2) x_1 + \cdots + N(\gamma_n, \sigma_n^2) x_n
= N(\gamma x, (\sigma^2 x^2),
\]
where the vectors of variances \( \sigma_i^2 \) and means \( \gamma_i \).
are denoted as \( \sigma^2 = (\sigma_1^2, \ldots, \sigma_n^2) \) and \( \gamma = (\gamma_1, \ldots, \gamma_n) \). The above calculations are similar, but the meaning is different, i.e. possibility vs. probability.

Since we deal with the space of fuzzy numbers, let us define the inclusion of fuzzy numbers.

**Definition 2.3.** The inclusion of fuzzy numbers with a degree \( 0 \leq h < 1 \), denoted as \( A_1 \supseteq_h A_2 \), is defined by \([A_1]_h \supseteq [A_2]_h\), where \([A]_h\) is \(h\)-level set of fuzzy number \(A\).

From Definition 2.3, \([A_1]_h \supseteq [A_2]_h\) is equivalent to

\[
\alpha_1 \leq \alpha_2 + |L^{-1}(h)| (c_1 - c_2),
\]

\[
\alpha_1 \geq \alpha_2 - |L^{-1}(h)| (c_1 - c_2).
\]

If \([A_1]_h \supseteq [A_2]_h\) for \(h \in (0, 1)\), we have:

\[
[A_1]_{h'} \supseteq [A_2]_{h'} \; \forall h' \leq h
\]

since \(c_1 \geq c_2\) and \(|L^{-1}(h')| \geq |L^{-1}(h)|\). It follows from (2.8) that the inclusion of fuzzy numbers with a degree \(h\) by Definition 2.3 holds at \(h' \leq h\).

### 3. Formulations of possibilistic linear regression

We propose three formulations of possibilistic linear regression by possibilistic linear systems (2.3) and discuss the identification of possibilistic parameters. The given data are represented as \((Y_i, \mathbf{x}_i)\), \(i = 1, \ldots, N\), where \(Y_i\) is a fuzzy observed value denoted by \(Y_i = (y_i, \epsilon_i)_L\) and \(\mathbf{x}_i = (x_{i1}, \ldots, x_{in})\) is a vector of explanatory variables for the \(i\)-th sample.

The basic concept for formulating three possibilistic linear regression is to use the mutual relation between the observed and estimated intervals which are obtained by \(h\)-level sets of the observed and the estimated fuzzy numbers respectively. Our idea is to obtain fuzzy parameters \(\hat{A}_i, \hat{A}_j\) and \(\hat{A}_i\) such that for \(i = 1, \ldots, N\),

\[
Y_i \subseteq_h \bar{Y}_i = \hat{A}_1 x_{i1} + \cdots + \hat{A}_n x_{in}, \tag{3.1}
\]

\[
Y_i \supseteq_h \bar{Y}_i = \hat{A}_1 x_{i1} + \cdots + \hat{A}_n x_{in}, \tag{3.2}
\]

\[
[Y_i]_h \cap [\bar{Y}_i = \hat{A}_1 x_{i1} + \cdots + \hat{A}_n x_{in}]_h \neq \emptyset. \tag{3.3}
\]

Equations (3.1) and (3.2) mean that all given outputs \(\{Y_i\}\) are covered by the estimated fuzzy numbers \(\{\bar{Y}_i\}\) and contain the estimated fuzzy numbers \(\{\tilde{Y}_i\}\), respectively. Furthermore, (3.3) implies that the intersection of the given output \(Y_i\) and estimated fuzzy number \(\tilde{Y}_i\) is not empty. The mutual relations given by (3.1)–(3.3) are illustrated in Figure 2. The formulations derived from (3.1)–(3.3) are called Min problem, Max problem and Conjunction problem, respectively.

**Min problem.**

\[
\text{Min}_{\hat{A}_j = (\bar{a}_j, \bar{c}_j)_L} \; \overline{J}(\bar{c}) = \sum \tilde{c} |x_i|
\]

subject to (3.1), where \(\tilde{c} |x_i|\) is the spread of estimated fuzzy output \(\bar{Y}_i\).

This problem can be interpreted as obtaining the smallest spread of \(\Sigma \bar{Y}_i\) such that \(\bar{Y}_i \supseteq_h Y_i\). Using (2.5) and Definition 2.3, Min problem can be reduced to the following linear programming:

\[
\text{Min}_{\hat{A}_j = (\bar{a}_j, \bar{c}_j)_L} \sum \tilde{c} |x_i| = \overline{J}(\bar{c}),
\]

\[
y_i + |L^{-1}(h)| |\epsilon_i| \leq \bar{a} x_i + |L^{-1}(h)| |\tilde{c}| |x_i|,
\]

\[
y_i - |L^{-1}(h)| |\epsilon_i| \geq \bar{a} x_i - |L^{-1}(h)| |\tilde{c}| |x_i|,
\]

\[
\tilde{c} \geq 0, \quad i = 1, \ldots, N.
\]

**Max problem.**

\[
\text{Max}_{\hat{A}_j = (\bar{a}_j, \bar{c}_j)_L} \; f(\bar{c}) = \sum \epsilon |x_i|
\]

subject to (3.2).

This problem can be interpreted as obtaining the largest spread of \(\Sigma \bar{Y}_i\) such that \(\bar{Y}_i \subseteq_h Y_i\). Using
(2.5) and Definition 2.3, Max problem can be reduced to the following linear programming:

\[
\text{Max } \sum_{i \in \mathcal{E}} \epsilon |x_i| = J(\epsilon), \\
y_i + |L^{-1}(h)| e_i \geq \alpha x_i + |L^{-1}(h)| \epsilon |x_i|, \\
y_i - |L^{-1}(h)| e_i \leq \alpha x_i - |L^{-1}(h)| \epsilon |x_i|, \\
\epsilon \geq 0, \quad i = 1, \ldots, N.
\]

Conjunction problem.

\[
\text{Min } J(\hat{c}) = \sum \hat{c} |x_i|
\]

subject to (3.3).

This problem can be interpreted as obtaining the smallest spread of \(\sum \hat{Y}_i\) such that \([Y_i]_h \cap \{\hat{Y}_i\}_h \neq \emptyset\). Using (2.5) and Definition 2.3, this problem becomes the following linear programming:

\[
\text{Min } \sum_{a, \hat{c}} \hat{c} |x_i| = J(\hat{c}), \\
y_i + |L^{-1}(h)| e_i \geq \hat{\alpha} x_i - |L^{-1}(h)| \hat{c} |x_i|, \\
y_i - |L^{-1}(h)| e_i \leq \hat{\alpha} x_i + |L^{-1}(h)| \hat{c} |x_i|, \\
\hat{c} \geq 0, \quad i = 1, \ldots, N.
\]

The optimal solutions of Min, Max and Conjunction problems are denoted as \(\hat{A}_j = (\hat{a}_j, \hat{c}_j)_L\), \(\hat{A}_j = (\hat{a}_j, \hat{c}_j)_L\) and \(\hat{A}_j = (\hat{a}_j, \hat{c}_j)_L\), respectively and the estimated fuzzy outputs are denoted as \(\hat{Y}_i\), \(\hat{Y}_i\) and \(\hat{Y}_i\), respectively.

In what follows, let us assume that the given data \((Y_i^0, x_i^0), i = 1, \ldots, N\) satisfy a possibilistic linear system:

\[
Y_i^0 = A_i x_i^0 + \cdots + A_n x_i^n, \quad i = 1, \ldots, N. \tag{3.7}
\]

Theorem 3.1. If \((Y_i^0, x_i^0), i = 1, \ldots, N\) satisfy (3.7), we obtain the following equalities by solving Min and Max problems.

\[
A^0 = \hat{A} = \hat{A}, \quad Y^0 = \hat{Y} = \hat{Y}. \tag{3.8}
\]

Proof. Let us prove only \(A^0 = \hat{A}\) in Min problem. Since \((Y_i^0, x_i^0)\) satisfies (3.7), we have \(e_i = c^0 |x_i^0|\). The constraint condition in (3.4) is

\[
y_i \leq \alpha x_i^0 + |L^{-1}(h)| (\hat{c} - c^0)|x_i^0|, \\
y_i \geq \alpha x_i^0 + |L^{-1}(h)| (c^0 - \hat{c})|x_i^0|. \tag{3.9}
\]

Thus, \((\alpha^0, c^0)\) is an admissible solution. Now it is assumed that there is \(\hat{c}'\) such that

\[
\sum \hat{c}' |x_i^0| < \sum c^0 |x_i^0|. \tag{3.10}
\]

There is some \(i\) for which

\[
\hat{c}' |x_i^0| < c^0 |x_i^0|. \tag{3.11}
\]

This is contradiction to (3.9). Hence \(\hat{c} = c^0\) is the minimum solution.

Substituting \(\hat{c} = c^0\) to (3.9), we have

\[
y_i^0 = \alpha x_i^0, \quad i = 1, \ldots, N. \tag{3.12}
\]

Thus, we have \(\alpha = \alpha^0\), since it is assumed generally that there are \(n\) independent vectors in \(\{x_1, \ldots, x_N\}\), where \(N > n\).

Theorem 3.2. If \((Y_i^0, x_i^0), i = 1, \ldots, N,\) satisfy (3.7), then \(J(\hat{c}) = 0\) in the Conjunction problem.

Proof. Substituting \(\hat{c} = 0\) to (3.6), the constraint condition is

\[
y_i^0 + |L^{-1}(h)| e_i \geq \hat{\alpha} x_i^0, \\
y_i^0 - |L^{-1}(h)| e_i \leq \hat{\alpha} x_i^0. \tag{3.13}
\]

Since \(\hat{c} = 0\) is an admissible solution, \(\hat{c} = 0\) is optimal, i.e. \(J(\hat{c}) = 0\). It should be noted that \(\hat{\alpha} = \alpha^0\) is one of solutions, but is not unique.

Theorem 3.3. The given data \((Y_i^0, x_i^0), i = 1, \ldots, N\) satisfy a conventional linear system \(Y_i^0 = a^0 x_i^0\), we have

\[
y_i = \hat{Y}_i = \hat{Y}_i = \hat{Y}_i, \quad a^0 = \hat{A} = \hat{A} = \hat{A}, \tag{3.14}
\]

Proof. Substituting \(c = 0\) to the constraint conditions of Min, Max and Conjunction problems, we obtain \(Y_i^0 = a^0 x_i^0\). Thus, we can obtain (3.14) by solving Min, Max and Conjunction problems.

4. Fuzzy data analysis

In general, the given data do not satisfy a possibilistic linear system (3.7). Thus, we have to discuss the existence of the solution and mutual properties in three problems.
Theorem 4.1. There are optimal solutions \( \mathbf{A} = (\mathbf{a}, \mathbf{c}) \) and \( \mathbf{A} = (\mathbf{a}, \mathbf{c}) \) for all \( h \in [0,1) \) in Min problem (3.4) and the Conjunction problem (3.6), but it is not assured that there exists an optimal solution in the Max problem (3.5).

Proof. Assumed \( \mathbf{a} = 0 \), the constraint condition in (3.4) yields
\[
-|L^{-1}(h)|(|e| x_i - e_i) 
\leq y_i \leq |L^{-1}(h)|(|e| x_i - e_i). \tag{4.1}
\]
If we take a sufficiently large number for all \( e_i \), \( \mathbf{A} = (0, e) \) is a feasible solution, because \( y_i \) is finite. Thus, there is an optimal solution in Min problem. The proof in the Conjunction problem can be done in the same way described above. Conversely, even when we take \( e_i = 0 \) in the constraint condition of the Max problem (3.5), it is not assured that there is an admissible set in (3.5). \( \square \)

Theorem 4.2. There exists an optimal solution in Max problem if and only if \( \hat{J}(e) = 0 \) in the Conjunction problem.

Proof. Assume that there is an optimal solution in Max problem. Taking \( e = 0 \) in (3.6), we have
\[
\hat{a} x_i \leq y_i + |L^{-1}(h)| e_i, \\
\hat{a} x_i \geq y_i - |L^{-1}(h)| e_i. \tag{4.2}
\]
Since it follows from the assumption that there is an admissible set of (3.5), there exists an admissible set of (4.2). Thus, \( \hat{e} = 0 \) is optimal solution in the Conjunction problem. Conversely assume that \( \hat{J}(e) = 0 \). It follows from (4.2) that there is an admissible set of (3.5) with \( e \geq 0 \). Thus there exists an optimal solution in the Max problem. \( \square \)

Theorem 4.3. \( \hat{J}(e) \geq J(h) \). \tag{4.3}

Proof. Let us denote the admissible sets of (3.4) and (3.6) as \( \hat{D} \) and \( \hat{D} \), respectively. From (3.4) and (3.6) we have
\[
y_i - |L^{-1}(h)| e_i \leq y_i + |L^{-1}(h)| e_i, \\
y_i + |L^{-1}(h)| e_i \geq y_i - |L^{-1}(h)| e_i, \tag{4.4}
\]
Since \( \hat{D} \subset \hat{D} \) from (4.4), \( \hat{J}(e) \geq J(h) \). \( \square \)

Theorem 4.4. If fuzzy outputs \( \mathbf{Y}_i = (y_i, e_i), i = 1, \ldots, N \), become crisp, i.e. \( e_i = 0, i = 1, \ldots, N \), we have
\[
\hat{Y}_i = Y_i. \tag{4.5}
\]

Proof. Substituting \( e_i = 0 \) to the Min problem (3.4) and the Conjunction problem (3.6), the constraint conditions of (3.4) and (3.6) are the same. Thus, \( \hat{Y}_i = Y_i \) since \( \hat{D} = \hat{D} \). \( \square \)

Now, we will discuss the given threshold \( h \). If we take a large value for \( h \), we consider only intervals of data which have high possibilities. Thus, this analysis is optimistic. Conversely, if we take a small value for \( h \), this analysis is pessimistic in the sense that we consider intervals of low possibilities. Although there are optimal solutions for all \( h \in [0,1) \) in Min and Conjunction problems, we have the following theorem with respect to the Max problem.

Theorem 4.5. If there is an optimal solution for \( h' \) in the Max problem, there exists an optimal solution for \( 0 \leq h \leq h' \) in the Max problem.

Proof. Denote the optimal solution for \( h' \) as \( (\mathbf{a}', e') \). Since \( h' \geq h \), it holds that \( |L^{-1}(h')| \leq |L^{-1}(h)| \). The constraint condition for \( h \) in the Max problem is rewritten as
\[
y_i + |L^{-1}(h')| e_i \geq y_i + |L^{-1}(h')| e_i \geq \mathbf{a}' x_i + |L^{-1}(h')| e' x_i, \\
y_i - |L^{-1}(h')| e_i \leq y_i - |L^{-1}(h')| e_i \leq \mathbf{a}' x_i - |L^{-1}(h')| e' x_i. \tag{4.6}
\]
The solution \( (\mathbf{a}', e' | L^{-1}(h') | / |L^{-1}(h)| ) \) satisfies (4.6). Since there is an admissible solution for \( h \), there is an optimal solution for \( h \) in the Max problem. \( \square \)

Let us denote the optimal solution and the performance index for \( h \) as \( (\mathbf{a}_h, e_h) \) and \( J(h) \), respectively.
Theorem 4.6. We have for $h' \leq h$
\[
\bar{J}(h') \leq \bar{J}(h), \quad (4.7)
\]
\[
\tilde{J}(h') \geq \tilde{J}(h), \quad (4.8)
\]
\[
\hat{J}(h') \leq \hat{J}(h). \quad (4.9)
\]

Proof. First, let us prove (4.7). The constraint conditions for $h$ are rewritten as
\[
y_i \leq \bar{a}_h x_i + |L^{-1}(h)| (\bar{c}_h |x_i| - e_i),
\]
\[
y_i \geq \bar{a}_h x_i - |L^{-1}(h)| (\bar{c}_h |x_i| - e_i).
\]

It follows from $\bar{Y} \subseteq_h Y$ that $\bar{c}_h |x_i| - e_i \geq 0$. Thus, we have
\[
y_i \leq \bar{a}_h x_i + |L^{-1}(h')| (\bar{c}_h |x_i| - e_i),
\]
\[
y_i \geq \bar{a}_h x_i - |L^{-1}(h')| (\bar{c}_h |x_i| - e_i).
\]

Therefore $(\bar{a}_h, \bar{c}_h)_L \subseteq \bar{D}_{h'}$ leads to $\bar{J}(h') \leq \bar{J}(h)$.

Next, let us prove (4.8). Similarly, we have
\[
y_i \geq \bar{a}_h x_i + |L^{-1}(h)| (\bar{c}_h |x_i| - e_i),
\]
\[
y_i \leq \bar{a}_h x_i - |L^{-1}(h)| (\bar{c}_h |x_i| - e_i).
\]

It follows from $\bar{Y} \supseteq_h Y$ that $\bar{c}_h |x_i| - e_i \leq 0$. Thus, we have
\[
y_i \geq \bar{a}_h x_i + |L^{-1}(h')| (\bar{c}_h |x_i| - e_i),
\]
\[
y_i \leq \bar{a}_h x_i - |L^{-1}(h')| (\bar{c}_h |x_i| - e_i).
\]

Therefore $(\bar{a}_h, \bar{c}_h) \subseteq \bar{D}_{h'}$ leads to $\tilde{J}(h') \geq \tilde{J}(h)$.

Lastly, it is easy to prove (4.9) in the same way. □

Theorem 4.7. If there is an optimal solution in the Max problem,
\[
\bar{J}(\bar{c}) \geq \tilde{J}(c) \quad \forall h.
\]

Proof. If there are optimal solutions in Min and Max problems, the following conditions must hold for all $i$:
\[
\bar{c} |x_i| \geq e_i, \quad \bar{c} |x_i| \leq e_i.
\]

Thus, we have
\[
\bar{J}(\bar{c}) = \min \sum \bar{c} |x_i| \geq \max \sum \bar{c} |x_i| = \bar{J}(\bar{c}),
\]
which leads to the theorem. □

In conclusion, Figure 3 can be obtained from Theorems 4.2–4.7. If fuzzy outputs are reduced to non-fuzzy outputs, the Conjunction problem is equivalent to the Min problem, although the Max problem does not exist.

5. Numerical examples

To illustrate our approach for dealing with fuzzy data, we will show two examples. The possibilistic linear model is
\[
Y_i = A_0 + A_1 x_{i1}, \quad i = 1, \ldots, N,
\]
where the fuzzy parameter $A_i$ is denoted as $A_i = (\alpha_i, \epsilon_i)_L$ and $L(x) = 1 - |x|$ is used. Thus, $|L^{-1}(h)| = 1 - h$.

Example 1. Consider the case where there is an optimal solution in the Max problem. Table 1 shows the given data. By solving the LP problems of (3.4), (3.5) and (3.6) with $h = 0$, the optimal solutions are obtained as

Min problem.
\[
\bar{A}_0 = (3.85, 3.85)_L, \quad \bar{A}_1 = (2.10, 0)_L,
\]
\[
\bar{J}(\bar{c}) = 19.25.
\]

<table>
<thead>
<tr>
<th>$Y_i = (Y_i, e_i)_L$</th>
<th>$x_{i1}$</th>
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<tbody>
<tr>
<td>(8,0,1.8)</td>
<td>1</td>
</tr>
<tr>
<td>(6.4,2.2)</td>
<td>2</td>
</tr>
<tr>
<td>(9.5,2.6)</td>
<td>3</td>
</tr>
<tr>
<td>(13.5,2.6)</td>
<td>4</td>
</tr>
<tr>
<td>(13,0,2.4)</td>
<td>5</td>
</tr>
</tbody>
</table>
Max problem.

A_0 = (4.63, 0)_L, A_1 = (1.78, 0.21)_L, \( J(\epsilon) = 3.13 \).

Conjunction problem.

\[ \hat{A}_0 = (4.63, 0)_L, \hat{A}_1 = (1.57, 0)_L, \hat{J}(\hat{\epsilon}) = 0. \]

The given fuzzy output \( Y_i \) and the estimated fuzzy outputs \( \bar{Y}_i, \hat{Y}_i, \hat{Y}_i \) are illustrated in Figure 4. Table 2 shows intervals of 0-level fuzzy outputs and estimated outputs. These results are consistent with the theorems described in 4.

Example 2. Consider the case where there is no optimal solution in the Max problem. Then, from Theorem 4.2 the optimal solution in the Conjunction problem is obtained as \( \hat{\epsilon} \geq 0 \). Table 3 shows the given data. By solving the LP problems of (3.4) and (3.6) with \( h = 0 \), the optimal solutions are obtained as

Min problem.

\[ \bar{A}_0 = (3.15, 4.55)_L, \bar{A}_1 = (2.10, 0)_L, \bar{J}(\hat{\epsilon}) = 22.75. \]

Conjunction problem.

\[ \hat{A}_0 = (4.35, 0.28)_L, \hat{A}_1 = (1.57, 0)_1, \hat{J}(\hat{\epsilon}) = 1.42. \]

\( Y_i, \bar{Y}_i, \hat{Y}_i \) are illustrated in Figure 5. Table 4 shows intervals of 0-level fuzzy outputs and estimated outputs.

### Table 2

<table>
<thead>
<tr>
<th>No.</th>
<th>Fuzzy output</th>
<th>Estimation outputs</th>
</tr>
</thead>
<tbody>
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<td></td>
<td>Min problem</td>
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<tr>
<td>1</td>
<td>[6.2, 9.8]</td>
<td>[2.1, 9.8]</td>
</tr>
<tr>
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<td>[4.2, 8.6]</td>
<td>[4.2, 11.9]</td>
</tr>
<tr>
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<td>[6.9, 12.1]</td>
<td>[6.3, 14.0]</td>
</tr>
<tr>
<td>4</td>
<td>[10.9, 16.1]</td>
<td>[8.4, 16.1]</td>
</tr>
<tr>
<td>5</td>
<td>[10.6, 15.4]</td>
<td>[10.5, 18.2]</td>
</tr>
</tbody>
</table>

### Table 3

Data in the numerical Example 2

\[ Y_i = (y_i, e_i)_L \quad x_{1i} \]

<table>
<thead>
<tr>
<th>No.</th>
<th>Fuzzy output</th>
<th>Estimation outputs</th>
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<tbody>
<tr>
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<td></td>
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<td>5</td>
<td>[10.6, 15.4]</td>
<td>[9.1, 18.2]</td>
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</tbody>
</table>
6. Concluding remarks

The conventional regression analysis based on probability models shows central tendency. From this viewpoint, some outlying data may be ignored as the 'bad' observation. In our case, it is assumed that all data have occurred for sure. We recognize that all data may be possible. Thus, we use a possibilistic linear systems as a model for analyzing the given data. Although our method can be applied to conventional data, our emphasis is on dealing with fuzzy outputs which are given in some cases by expert knowledge and also in some cases by several times of observation.

References